Solution to Assignment 3

Supplementary Problems

1. Express the straight line $ax+by=1,\ a,b>0$, in polar coordinates. How about $ax+by=0$?

Solution. Let $c=\sqrt{a^2+b^2}$. Equation is

$$1=r(a\cos\theta+b\sin\theta)=rc\left(\frac{a}{c}\cos\theta+\frac{b}{c}\sin\theta\right)=rc\sin(\theta+\alpha),$$

where α satisfies $\sin\alpha=a/c$. In polar coordinates, the straight line is given by

$$r=\frac{1}{c\sin(\theta+\alpha)}\ ,\quad \theta\in(-\alpha,-\alpha+\pi)\ .$$

2. Express the hyperbola $x^2-y^2=1$ $(y\geq 0)$ in polar coordinates.

Solution. From $1=r^2(\cos^2\theta-\sin^2\theta)=r^2\cos 2\theta$ we get

$$r=\frac{1}{\sqrt{\cos 2\theta}}\ ,$$

where $\theta\in(-\pi/4,\pi/4)$.

3. Discuss the existence of the improper integral

$$\iint_D \frac{y}{(x^2+y^2)^{3/2}}\ ,$$

where D is the region enclosed by the polar graph $r=1+\cos\theta$.

Solution. Let a be a small positive number. We consider

$$I(a)\equiv\iint_{D_a}\frac{y}{(x^2+y^2)^{3/2}}\ ,$$

where D_a is the region bounded between $r=1+\cos\theta, y\geq 0$ and $r=a$. Using polar coordinates,

$$I(a)=\int_0^{\theta_0}\int_a^{1+\cos\theta}\frac{r\sin\theta}{r^3}r\,drd\theta\ ,$$

where θ_0 satisfies $1+\cos\theta_0=a$. Hence

$$I(a)=\int_0^{\theta_0}\sin\theta(\log(1+\cos\theta)-\log a)\,d\theta\ .$$

On one hand, we have

$$\begin{aligned}\int_0^{\theta_0}\sin\theta\log(1+\cos\theta)\,d\theta &= -\int_1^{\cos\theta_0}\log(1+t)\,dt\\ &= -(1+\cos\theta_0)\log(1+\cos\theta_0)+\cos\theta_0+2\log 2-1\\ &\to 2\log 2-2\ .\end{aligned}$$

as $a\to 0$ $(\theta_0\to\pi)$. On the other hand,

$$-\int_0^{\theta_0}\sin\theta\log a\,d\theta=\log a(\cos\theta_0-1)\to-\infty$$

as $a\to 0$. That is,

$$\lim_{a\to 0}I(a)=-\infty\ ,$$

the improper integral does not exist.

MATH2020B HW3 Solution.

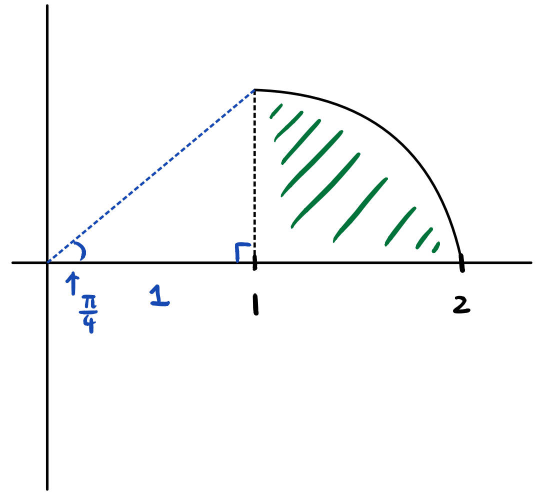
Q22:

$$\int_1^2 \int_0^{\sqrt{2x-x^2}} \frac{1}{(x^2+y^2)^2}\, dy\, dx$$

$$= \int_0^{\pi/4} \int_{\frac{1}{\cos\theta}}^{2\cos\theta} \frac{1}{r^4}\, r\, dr\, d\theta$$

$$= (\text{ Steps } \dots)$$

$$= \frac{\pi}{16}$$



① $2x - x^2 = y^2$
$x^2 + y^2 - 2x = 0$ $\Rightarrow$ $r^2 - 2r\cos\theta = 0$
$r = 2\cos\theta$
$\Rightarrow (x-1)^2 + y^2 = 1$

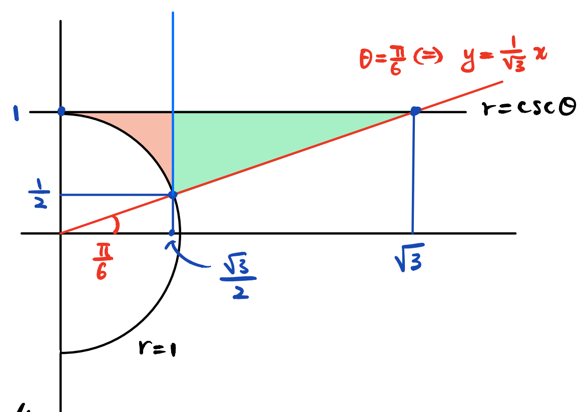
② $0 \le \theta \le \frac{\pi}{4}$, $\frac{1}{\cos\theta} \le r \le 2\cos\theta$

Q24

$$\int_{\pi/6}^{\pi/2} \int_1^{\csc\theta} r^2 \cos\theta\, dr\, d\theta$$

$$\left[\begin{aligned} & r^2 \cos\theta\, dr\, d\theta \\ & = (r\cos\theta)\, r\, dr\, d\theta \end{aligned} \right]$$

$\Downarrow$



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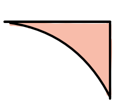
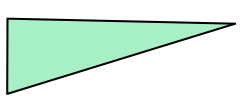
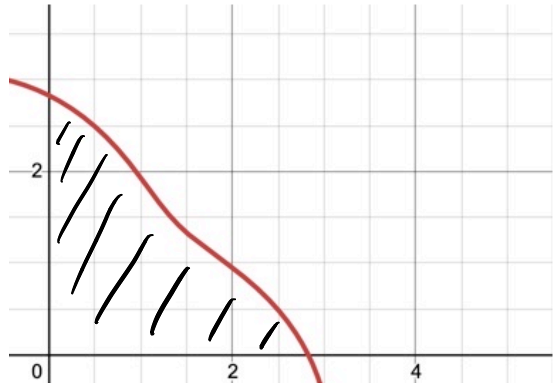
 +  $= \displaystyle\int_0^{\sqrt{3}/2} \int_{\sqrt{1-x^2}}^{1} x\, dy\, dx + \int_{\frac{\sqrt{3}}{2}}^{\sqrt{3}} \int_{\frac{x}{\sqrt{3}}}^{1} x\, dy\, dx$

Q27

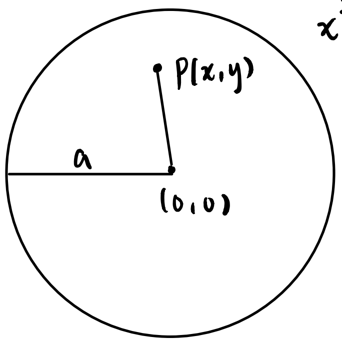
$$\int_0^{\frac{\pi}{2}} \int_0^{2(2-\sin 2\theta)^{\frac{1}{2}}} r \, dr \, d\theta$$



$= (\text{Steps} \cdots)$

$= 2\pi - 2.$

Q35



$x^2+y^2 = a$

$D := \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \leq a^2\}$

$$\text{Averge distame} = \frac{1}{\pi a^2} \iint_D \sqrt{x^2+y^2} \, dA$$

$$= \frac{1}{\pi a^2} \int_0^{2\pi} \int_0^a r^2 \, dr \, d\theta$$

$$= \frac{1}{\pi a^2} \cdot \frac{2\pi a^3}{3}$$

$$= \frac{2a}{3}.$$